# ON A MIXED BOUNDARY VALUE PROBLEM FOR THE WAVE EQUATION

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*PMM Vol.22, No.6, 1958, pp. 829-832*

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*(Received 15 July 1957)*

The first and the second boundary value problems

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial t^2} = 0 \tag{1}$$

for the semi-plane $y > 0$ are well explored [1]. The mixed boundary value problem is somewhat more difficult:

$$\left.\begin{aligned} &u(x, t) = 0 \qquad \text{for } |x| > 1 \\ &w(x, t) \equiv \frac{\partial u}{\partial y} = a(x, t) \qquad \text{for } |x| < 1 \end{aligned}\right\} \quad \text{when } y = 0 \tag{2}$$

Here $a(s, t)$ is a bounded function.

For simplicity, the initial conditions are assumed equal to zero:

$$u = \partial u / \partial t = 0 \qquad \text{at } t = 0$$

If $u(x, y, t)$ is taken as the displacement potential of an elastic fluid, the boundary conditions may have the following meaning: a punch of assigned form $a(x, t)$ is pressed into the boundary of a semi-plane in the interval $-1 < x < 1$. The remaining part of the boundary is free of pressure.

The purpose of this paper is a construction of the values of the functions $u(x, t)$ on the boundary ($y = 0$) for $|x| < 1$ and $w(x, t)$ for $|x| > 1$. When these are known, the function $u(x, y, t)$ can be constructed in the entire semi-plane $y > 0$, $t > 0$.

As known, the functions $u(x, t)$ and $w(x, t)$ are related by the expression [1]

$$u(x,\ t) + \frac{1}{\pi} \iint\limits_{\sigma} \frac{w(\xi,\ \tau)\, d\xi\, d\tau}{\sqrt{(t-\tau)^2 - (x-\xi)^2}} = 0 \tag{3}$$

The domain of integration is shown in Fig. 1.

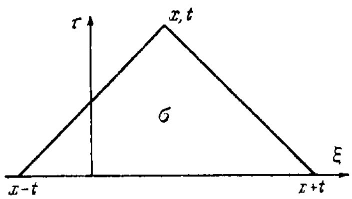


Fig. 1.

The following auxiliary problem is discussed first:

The function $w(x,\ t)$ is to be determined, using (3) for $x > 0$, $t > 0$, under condition that

$$w(x,\ t) = a(x,\ t) \quad \text{for } x < 0, \quad u(x,\ t) = 0 \quad \text{for } x > 0 \tag{4}$$

The relationship (3) for the determination of $w(x,\ t)$ gives for $x > 0$ the nonhomogeneous integral equation of the first kind

$$\frac{1}{\pi} \iint\limits_{\sigma} \frac{w(\xi,\ \tau)\, d\xi\, d\tau}{\sqrt{(t-\tau)^2 - (x-\xi)^2}} = 0, \qquad x > 0 \tag{5}$$

A solution is to be found, which would be limited for $x > 0$, $t \geqslant 0$ and could be integrated with respect to $x$ in every finite interval for any $t$.

Multiply both parts of this equation by $e^{-pt}$ and integrate with respect to $t$ from zero to infinity, taking into account that Re $p > 0$. This results in:

$$\int_0^\infty K_0(p\,|x-\xi|)\, \varphi(\xi,\ p)\, d\xi + \int_{+\infty}^{0} K_0(p\,|x-\xi|)\, \psi(\xi,\ p)\, d\xi = 0 \tag{6}$$

where $K_0(\xi)$ is a MacDonald function.

$$\varphi(x,\ p) = \int_0^\infty w(x,\ t)\, e^{-pt} dt \quad \text{for } x > 0, \qquad \psi(x,\ p) = \int_0^\infty a(x,\ t)\, e^{-pt} dt \quad \text{for } x < 0 \tag{7}$$

The integral equation (6) is to be solved by Fok's method [ 2 ]. Both parts of (6) are multiplied by $e^{-sx}$, where Re $s > 0$, and integrated with respect to $x$ from zero to infinity. Calculations analogous to those in paper [ 2 ] result in

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Phi(\eta,p)\,d\eta}{\sqrt{p^2-\eta^2}\,(\eta-s)}+F(s,p)=0 \qquad (0<\operatorname{Re}s<\operatorname{Re}p) \tag{8}$$

where

$$\Phi(s,p)=\int_0^\infty \varphi(\xi,p)\,e^{-s\xi}d\xi$$

$$F(s,p)=\int_p^{(p\infty)}\frac{\Psi(\zeta,p)\,d\zeta}{(\zeta+s)\sqrt{\zeta^2-p^2}},\qquad \Psi(s,p)=\int_{-\infty}^{0}e^{s\xi}\psi(\xi,p)\,d\xi \tag{9}$$

The function $a(x, t)$ must satisfy the Dirichlet condition. Then $\Psi(s, p)$ will be a regular function of a complex variable $s$ for $\operatorname{Re} s > 0$. Obviously, then $F(s, p)$ will be also a regular function for $\operatorname{Re} s > 0$. It can be expressed by the Cauchy integral

$$F(s,p)=-\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{F(\eta,p)\,d\eta}{\eta-s}$$

Expression (8) can be rewritten as

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{H(\eta,p)\,d\eta}{\eta-s}=0 \qquad (0<\operatorname{Re}s<\operatorname{Re}p) \tag{10}$$

where

$$H(s,p)=\frac{\Phi(s,p)}{\sqrt{p^2-s^2}}+F(s,p) \tag{11}$$

According to our remarks about the function $w(x, t)$, the function $\Phi(s, p)$ must be regular for $\operatorname{Re} s > 0$, and must also approach zero as $s$ tends to infinity. The function $F(s, p)$ is regular for $\operatorname{Re} s > 0$, as mentioned above. Repeating the reasoning of paper [2], we conclude from (10) that the function $H(s, p)$ must be regular for $\operatorname{Re} s < \operatorname{Re} p$.

Thus, the following problem leads to the determination of $\Phi(s, p)$:

Find a function $\Phi(s, p)$, regular for $\operatorname{Re} s > 0$ and approaching zero as $s$ tends to infinity, such that the function $H(s, p)$ is regular for $\operatorname{Re} s < \operatorname{Re} p$.

It is not difficult to verify that a solution of this problem is

$$\Phi(s,\ p) = -\sqrt{p+s}\int_{p}^{p\infty}\frac{\Psi(\zeta,\ p)\,d\zeta}{(\zeta+s)\sqrt{\zeta-p}} \tag{12}$$

A solution of the equation (5) is obtained after inverse transformation:

$$w(x,\ t) = 0 \quad \text{for } x > t$$
$$w(x,\ t) = -\frac{1}{\pi}\int_{x}^{t} a(x-\tau,\ t-\tau)\sqrt{\frac{\tau}{x}-1}\,\frac{d\tau}{\tau} \quad \text{for } x < t \tag{13}$$

Substituting the value $\Phi(s,\ p)$ from (12) into (11) and performing with (11) similar transformations in reverse order, we will show that the function $w(x,\ t)$ from (13) is indeed the solution of the integral equation (5). Uniqueness of solution of the equation (5), on our assumptions, can easily be shown if we investigate the homogeneous equation in a similar way.

Substituting the value $w(x,\ t)$ into equation (3), we obtain the expression $u(x,\ t)$ for $x < 0$.

The auxiliary problem is thus solved. Let us now show how the functions $w(x,\ t)$ for $|x| > 1$ and $u(x,\ t)$ for $|x| < 1$ can be determined for any instant of time, using the auxiliary problem, and the formula (3) for the boundary conditions (2). Let us investigate the plane $(xt)$.

In the domains $S_{01}$ and $S_{02}$ the vanishing initial values give $w = 0$. In the domains $S_{11}$ and $S_{12}$ the function $w(x,\ t)$ is known from the solution of our auxiliary problem. The value of $u(x,\ t)$ in the domain $P_{00}$ is given by the formula (3), and in the domains $P_{11}$ and $P_{12}$ by the same formula, since the value of $w$ in the domains $S_{11}$ and $S_{12}$ is known. In the domains $S_{21}$ and $S_{22}$ the function $w(x,\ t)$ again results from the solution of the auxiliary problem, since the value $u(x,\ t)$ in the domains $P_{00}$, $P_{11}$ and $P_{12}$ is already known (Fig. 2).

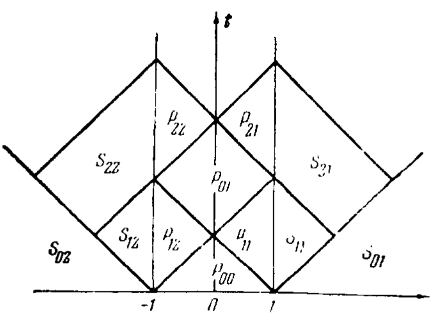


Fig. 2.

The value $u(x, t)$ in the domains $P_{01}$, $P_{21}$ and $P_{22}$ can also be constructed, taking into account the already known values of $w(x, t)$. Continuing this process further we can construct the values of $w(x, t)$ for $|x| > 1$ and $u(x, t)$ for $|x| < 1$ for any instant of time, as required.

A solution is thus obtained in principle.

Several remarks are due with regard to the features of this solution. Let us study the solution of the auxiliary problem. It follows from the formula (13) that for $x = t$ on the front of the propagating wave

$$w = \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} = 0$$

In the vicinity of the point $x = 0$ we have for $a(x, t) = 1$

$$w(x, t) = \frac{2}{\pi}\left[\cos^{-1}\sqrt{\frac{x}{t}} - \sqrt{\frac{t}{x} - 1}\right], \qquad \frac{t}{x} > 1$$

In the vicinity of $x = 0$ the function $w(x, t)$ approaches infinity, as $1/\sqrt{x}$. It is noteworthy that the function $w(x, t)$ has this property for all $a(x, t)$, except those for which

$$\int_0^t a(-\tau, t-\tau)\frac{d\tau}{\sqrt{\tau}} = 0 \tag{14}$$

for any $t$. It is not difficult to verify the fact that the function $a(x, t)$ will satisfy the condition (14) if the value of $\partial u/\partial y$ is taken instead of $a(x, t)$ at $y = 0$ and $x < 0$, corresponding to

$$u(x, t) = b(x, t) \quad \text{for } x < 0, \qquad u(x, t) = 0 \quad \text{for } x > 0$$

$$b = 0, \qquad \frac{\partial b}{\partial x} = 0 \qquad \text{at} \quad x = 0$$

for example, $b(x, t) = c(t)x^2$. This $a(x, t)$ will satisfy condition (14).

As known [3], an analogous mixed problem for Laplace's equation leads to the values of $\partial u/\partial y$ at $y = 0$, which have the same features as here, at points where the type of boundary conditions changes. There will be no singularity if the relationship of type (14) is fulfilled for the boundary value.

The following problem may be investigated analogously: at $s = 0$

$$u(x, t) = a(x, t) \quad \text{for } |x| < 1, \qquad \frac{\partial u}{\partial y} = 0 \quad \text{for } |x| > 1 \tag{15}$$

Let us investigate the elastic semi-space $y > 0$.

If $u(x, y, t)$ is the displacement in transverse oscillations of the elastic semi-space, polarized parallel to the $z$-axis, then $\partial u/\partial y = \tau_{zy}/\mu$, where $\tau_{zy}$ is the shear stress, $\mu$ is the shear modulus.

Boundary conditions (15) can be interpreted in this way: displacements $u = a(x, t)$ are given at the boundary of the elastic semi-space $y \geqslant 0$, in a strip $|x| < 1$; these displacements are parallel to the $z$-axis, independent from the $z$ coordinate; the remaining part of the boundary $|x| > 1$ is stress free. Then, as known, the problem becomes two-dimensional, and the system of equations representing the vibrations of elastic space degenerates into one wave equation for the component $u$, which is parallel to the $z$-axis.

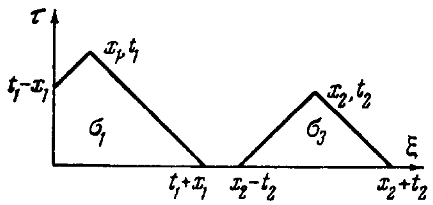


Fig. 3.

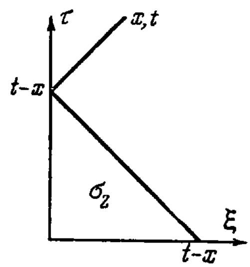


Fig. 4.

The solution of the auxiliary problem with the boundary conditions at $y = 0$

$$u(x, t) = a(x, t) \quad \text{for } x > 0, \qquad \frac{\partial u}{\partial y} \equiv w = 0 \quad \text{for } x < 0$$

will be

$$w(x, t) = \frac{1}{\pi}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)\iint_{\sigma_1} \frac{a(\xi, \tau)\, d\xi\, d\tau}{\sqrt{(t-\tau)^2 - (x-\xi)^2}} -$$

$$- \frac{1}{\tau}\,\frac{\partial^2}{\partial t^2}\iint_{\sigma_2} R\left(\frac{t-\tau}{x}, \frac{x}{\xi}\right) a(\xi, \tau)\, d\xi\, d\tau \qquad \text{when } x < t$$

$$w(x, t) = \frac{1}{\pi}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right)\iint_{\sigma_3} \frac{a(\xi, \tau)\, d\xi\, d\tau}{\sqrt{(t-\tau)^2 - (x-\xi)^2}} \qquad \text{when } x > t$$

Here

$$R\left(\frac{t-\tau}{x}, \frac{x}{\xi}\right) = -\frac{1}{\xi - x}\sqrt{\frac{x}{\xi}}\left[\frac{t-\tau-\xi}{2x} + \frac{1}{2} + \frac{t-\tau}{\xi - x} - \sqrt{\frac{(t-\tau)^2}{(\xi - x)^2} - 1}\right]$$

and the domains of integration $\sigma_1$, $\sigma_2$, $\sigma_3$ are shown in Figs. 3 and 4.

Using the formula which expresses $\partial u/\partial y \equiv w$ at $y = 0$ in terms of $u$ at $y = 0$, and which was derived in [1],

$$w = \frac{1}{\pi}\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\iint\limits_{\sigma} \frac{u(\xi, \tau)\, d\xi\, d\tau}{\sqrt{(t-\tau)^2 - (x-\xi)^2}}$$

by similar transformations it is possible to construct $w(x, t)$ for $|x| < 1$ and $u(x, t)$ for $|x| > 1$. The domain of integration for $\sigma$ is shown in Fig. 1.

It is important to mention that the two-dimensional case of the mixed boundary value problem (a punch without friction) can be investigated in a similar way for dynamic equations of theory of elasticity.

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